1. Review:

1.1 拓扑(开集)　　一般用花体字母表示集合的集合

$(X, \mathcal{T})$

O1: $X, \phi \in \mathcal{T}$

O2: $G_\alpha \in \mathcal{T} \Rightarrow \bigcup_\alpha G_\alpha \in \mathcal{T}$

O3: $G_1, G_2 \in \mathcal{T} \Rightarrow G_1 \cap G_2 \in \mathcal{T}$

例. (1) Sorgenfrey

$\mathbb{R}, \mathcal{T}_{Sorgenfrey} = \{U \subset \mathbb{R}: \forall x \in U, \exists \varepsilon > 0, s.t. [x, x+\varepsilon) \subset U\}$

(2) $\mathcal{T}_{cocountable}$

$\mathbb{R}, \mathcal{T}_{cocountable} = \{U \subset \mathbb{R}: U^c$ 至多可数$\}$

(3) $\mathcal{T}_{discrete}$, $\mathcal{T}_{trivial}$

(4) (X, d) metric space.

$\mathcal{T}_d = \{U \subset X \mid \forall x \in U, \exists \varepsilon > 0, B(x, \varepsilon) \subset U\}$

(X, d) 自然蕴含 $(X, \mathcal{T}_d)$

1.2 邻域.

拓扑空间 $(X, \mathcal{T})$, $x \in X$,

$\mathcal{N}(x) = \{N \subset X \mid x \in X, \exists U \in \mathcal{T}, x \in U \subset N\}$

例. $[0,1)$ 是 0 的邻域 in $\mathcal{T}_{Sorgenfrey}$, 但在 $\mathcal{T}_{Euclid}$ 中不是

1.3 闭集

$(X, \mathcal{T})$, F is closed $\Leftrightarrow F^c \in \mathcal{T}$

抽象成公理形式的定义: (先定义闭集, 再定义拓扑

集合 X, $\mathcal{C} \subset \mathcal{P}(X)$, 若

C1: $X, \phi \in \mathcal{C}$

C2: $\forall F_\alpha \in \mathcal{C}$, $\bigcap_\alpha F_\alpha \in \mathcal{C}$

C3: $F_1, F_2 \in \mathcal{C}$, $F_1 \cup F_2 \in \mathcal{C}$

1.4. 内部

$\mathrm{Int}\, A = \{a \in A \mid \exists G_a \in \mathcal{T},\ a \in G_a \subset A\}$

$= \{a \in A \mid A \in \mathcal{N}(a)\}$

1.5. 拓扑基. Basis

回忆 $\mathcal{T}_{Sorgenfrey}$, $\mathcal{T}_{Euclid}$, $\mathcal{T}_d$ 的定义, 不难看出

$$\mathcal{T} = \{U \mid \forall x \in U,\ \exists B_x,\ x \in B_x \subset U\}$$

如何定义这样的 B_x? 记这样的 B_x 全体为 $\mathcal{B}$

(1) $\forall x$, $\exists B_x \ni x \Rightarrow \bigcup_{B \in \mathcal{B}} B = X$ (B1)

(2) 两开集的交为开集 $\Rightarrow$ 若 $x \in B_\alpha$, $x \in B_\beta$, 则 $\exists B_\gamma$, s.t.
$x \in B_\gamma \subset B_\alpha \cap B_\beta$. (B2)

若 $\mathcal{B}$ 满足 (B1) (B2) 称为 Base,

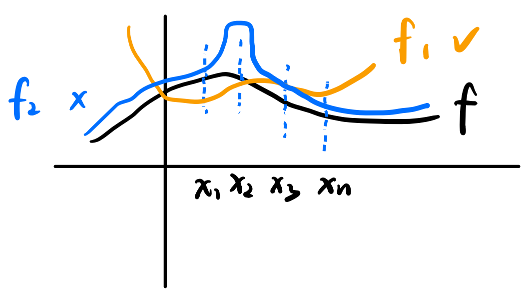
$$\mathcal{T} = \{U \mid \forall x \in U,\ \exists B_x \in \mathcal{B},\ x \in B_x \subset U\}$$

为 $\mathcal{B}$ 诱导的拓扑.

2. 拓扑空间的例 $\mathbb{R}^n$, 2^X ...

例. X 是集合, (Y, d) metric space, $Y^X = \{f : X \to Y\}$

$$\mathcal{B} = \{w(f; x_1, x_2, \cdots x_n; \varepsilon) : f \in Y^X,\ n \in \mathbb{N},\ x_1, \cdots, x_n \in X,\ \varepsilon > 0\}$$



$w(f; x_1, x_2, \cdots x_n; \varepsilon)$

$= \{g \in Y^X : \forall i = 1, \cdots, n,\ d(g(x_i), f(x_i)) < \varepsilon\}$

$\mathcal{T}_{p.c}$

逐点收敛拓扑.

若 $f_n \to f$, $\forall x$, 显然 依度量收敛

若 $f_n \to f$, $\forall x \neq x_0$, $f_n \not\to f$, $x = x_0$, $f_n \notin w(f; x_0; \varepsilon)$

Fact. 在这个拓扑下，$\{f_n\}$ 极限存在，必唯一.

例. $f_n = x^n$ on $[0,1]$，$f_n \to f = \begin{cases} 0, & [0,1) \\ 1, & \{1\} \end{cases}$

连续函数极限未必连续！

Q1: 有没有一种保持函数连续性的拓扑？

A1: (Y^X, d_∞)，$(X, \mathcal{T})$，$f: X \to Y$ continuous

$d_\infty(f,g) = \max_{x\in X} d(f(x), g(x))$

Rmk: (Y,d)，X 仅为一般"拓扑空间"

(Y,d)，X 仅为"集合"

$\mathcal{T}_{u.c} = \{U \mid \forall f\in U, \exists w(f;X;\varepsilon) \subset U\}$

以下方便起见，$X\subset\mathbb{R}$，$Y=\mathbb{R}$，$\mathcal{T}_X = \mathcal{T}_Y = \mathcal{T}_{Euclid}$

Prop. $f_n \xrightarrow{d_\infty} f$，$f_n: (X,d) \to (\mathbb{R},d)$ cts (continuous) $\Rightarrow f$ cts.

Pf:

Prove: $\forall x\in X$，f is cts at x.

$\forall \varepsilon>0$，$\exists n$，$d_\infty(f_n, f) < \varepsilon/3$

$\exists \delta>0$，$\forall |y-x|<\delta$，$|f_n(y)-f_n(x)| < \varepsilon/3$

$\Rightarrow |f(y)-f(x)| < \varepsilon$

Q2: $X = [0,1)$，$f_n(x) = \sum_{i=0}^{n} x^i$ （这里认为 $0^0=1$）

$f_n(x) \to \frac{1}{1-x}$，但并不一致.

"内闭一致" 如何刻画？

A2:

$w(f;F;\varepsilon)$，其中 $F\subset[0,1)$ 为任意闭集 (in fact, compact)

$\mathcal{T}_{c.c.} = \{U\in\mathbb{R}^{[0,1)} \mid \forall f\in U, \exists F$ closed, $\varepsilon>0$, s.t. $f\in w(f;F;\varepsilon)\subset U\}$

Prop. $f_n: [0,1) \to \mathbb{R}$, $f_n \to f$ in $\mathcal{T}_{c.c.}$ 则 f cts.

Pf. $\forall x \in [0,1)$, $\exists F$ 闭, $x \in F \subset [0,1)$

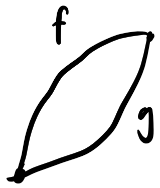
f_n 限制在 F 上一致收敛，故 f 在 F 上 cts，在 x 处 cts.

3. 再来一些度量空间的例子.

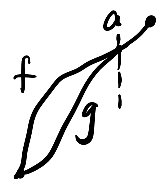
例. $X = \{f: [0,1] \to [0,1]$, cts, $f(0)=0$, $f(1)=1\}$

$d(f,g) = \sup \{r \mid f(r) \neq g(r)\} \cup \{0\}$

证其为 metric.



$d(f,g) = 1$

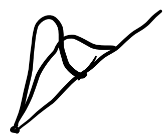


$d(f,g) = a$

$d(f,g) = 0 \iff \forall r, f(r) = g(r) \iff f = g$

$d(f,g) + d(g,h) = r_1 + r_2$

$\forall r > \max\{r_1, r_2\}$, $f(r) = g(r) = h(r) \Rightarrow d(f,h) \le \max\{r_1, r_2\}$

收敛方式：

例 $\mathbb{R}^{\mathbb{N}}$

$\boxed{\begin{array}{l} X = \mathbb{N} \\ Y = \mathbb{R} \\ \mathcal{T}_{u.n} \end{array}}$

$d_p((x^n),(y^n)) = (\sum_{n=1}^{\infty} |x^n - y^n|^p)^{1/p}$ $\quad 1 \le p < \infty$

$l_p = \{(x^n) \in \mathbb{R}^{\mathbb{N}} \mid d_p((x^n), 0) < \infty\}$

$d_\infty((x^n),(y^n)) = \sup_n |x^n - y^n|$

$l_\infty = \{(x^n) \in \mathbb{R}^n \mid d_\infty((x^n), 0) < \infty\}$

$d_p((x^n),(y^n)) = \sum_{n=1}^{\infty} |x^n - y^n|^p$ $\quad 0 < p \le 1$

$l_p = \{(x^n) \in \mathbb{R}^{\mathbb{N}} \mid d_p((x^n), 0) < \infty\}$

$d_H = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x^n - y^n|}{1 + |x^n - y^n|}$

$\mathcal{T} = \{U_n \times \mathbb{R}^{\mathbb{N}} \mid n \in \mathbb{N}, U_n$ 是 $\mathbb{R}^n$ 中开集$\}$

证明：$\mathcal{J} = \mathcal{J}_{d_H}$

- $\forall\, U \in \mathcal{J}_{d_H}$, $\forall\, x \in U$, $\exists\, \varepsilon > 0$, $B_H(x, \varepsilon) \subset U$.

  $\exists\, N > 0$, $\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2}$,

  $d_H^n = \sum_{k=1}^{n} \frac{1}{2^k} \frac{|x^k - y^k|}{1+|x^k-y^k|}$ 是 $\mathbb{R}^n$ 中度量，

  可验证 $B_H^n((x^k), \varepsilon)$ 为 $(\mathbb{R}^n, \mathcal{J}_{Euclid})$ 中开集，进而

  $B_H^n((x^k), \frac{\varepsilon}{2}) \times \mathbb{R}^{\mathbb{N}} \in \mathcal{J}$,

  $(x^k) \in B_H^n((x^k), \frac{\varepsilon}{2}) \times \mathbb{R}^{\mathbb{N}} \subset U \Rightarrow U \in \mathcal{J}$

- $\forall\, U \in \mathcal{J}$, $\forall\, x \in U$, $\exists\, n$, $U_n \subset \mathbb{R}^n$ 开，$x \in U_n \times \mathbb{R}^{\mathbb{N}} \subset U$

  $\exists\, \varepsilon_1 \cdots \varepsilon_n$, $x \in (x^1-\varepsilon_1, x^1+\varepsilon_1) \times \cdots \times (x^n - \varepsilon_n, x^n+\varepsilon_n) \times \mathbb{R}^{\mathbb{N}} \subset U$

  取 ε 充分小, s.t. $\varepsilon < \inf_{i=1,\dots,n} \{ \frac{n}{2^i} \frac{\varepsilon_i}{1+\varepsilon_i} \}$

  则 $B_H((x^k), \varepsilon) \subset (x^1-\varepsilon_1, x^1+\varepsilon_1) \times \cdots \times (x^n-\varepsilon_n, x^n+\varepsilon_n) \times \mathbb{R}^{\mathbb{N}}$

  (因 $\forall\, y^k \notin$ R.H.S. $\exists\, i$, $|y^i - x^i| \geq \varepsilon_i$,

  $$\sum \frac{1}{2^k} \frac{|x^k-y^k|}{1+|x^k-y^k|} \geq \frac{1}{2^i} \frac{|x^i-y^i|}{1+|x^i-y^i|} > \varepsilon\ )$$

  故 $U \in \mathcal{J}_H$